

MEI Core 4

Differential equations

Chapter assessment

1. Find y in terms of x given that $\frac{dy}{dx} = x(y-1)$. [4]
2. Solve $(x-1)\frac{dy}{dx} = xy$ for $x > 1$ and $y > 0$, given that $y = 1$ when $x = 3$. [5]
3. Obtain a particular solution to $(1-e^{2y})\frac{dy}{dx} = e^y$ given that $y = 0$ when $x = 2$.
(There is no need to express y in terms of x). [5]
4. Find an expression for y in terms of x given that $x^2\frac{dy}{dx} - y^2 = 0$. [4]
5. At time t seconds the rate of increase in the concentration of flesh eating bugs in a controlled environment is proportional to the concentration C of bugs present. Initially $C = 100$ bugs and after 2 seconds there are five times as many.
 - (i) Write down a differential equation connecting $\frac{dC}{dt}$, C and t and hence find an expression for C in terms of t . [7]
 - (ii) How many bugs are present after 5 seconds? [2]
 - (iii) When will the number of bugs exceed 5000? [3]
 - (iv) Find the time at which the concentration of bugs has increased by 50% of the initial concentration. [3]
6. Water is pouring out of a small hole in the bottom of a conical container of height 25 cm. Initially the container is full.
The rate of change of the height x of the water remaining in the container is given by
$$\frac{dx}{dt} = -\frac{50}{\pi}x^{-\frac{3}{2}}.$$
 - (i) Solve the differential equation to find x in terms of t . [5]
 - (ii) How long does it take for the container to empty completely? [2]

Total 40 marks

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Solutions to Chapter assessment

1. $\frac{dy}{dx} = x(y-1)$

$$\int \frac{1}{y-1} dy = \int x dx$$

$$\ln|y-1| = \frac{1}{2}x^2 + c$$

$$y-1 = e^{\frac{1}{2}x^2+c} = Ae^{\frac{1}{2}x^2}$$

$$y = 1 + Ae^{\frac{1}{2}x^2}$$

Replacing e^c with A

[4]

2. $(x-1)\frac{dy}{dx} = xy$

$$\int \frac{1}{y} dy = \int \frac{x}{x-1} dx = \int \left(\frac{x-1}{x-1} + \frac{1}{x-1} \right) dx = \int \left(1 + \frac{1}{x-1} \right) dx$$

$$\ln y = x + \ln(x-1) + c$$

$$\ln\left(\frac{y}{x-1}\right) = x + c$$

$$\text{When } x = 3, y = 1 \Rightarrow \ln\left(\frac{1}{2}\right) = 3 + c \Rightarrow c = \ln\left(\frac{1}{2}\right) - 3 = -3 - \ln 2$$

$$\ln\left(\frac{y}{x-1}\right) = x - 3 - \ln 2$$

$$\ln\left(\frac{2y}{x-1}\right) = x - 3$$

$$\frac{2y}{x-1} = e^{x-3}$$

$$y = \frac{(x-1)e^{x-3}}{2}$$

[5]

3. $(1-e^{2y})\frac{dy}{dx} = e^y$

$$\int (1-e^{2y})e^{-y} dy = \int 1 dx$$

$$\int (e^{-y} - e^y) dy = \int 1 dx$$

$$-e^{-y} - e^y = x + c$$

$$\text{When } x = 2, y = 0 \Rightarrow -1 - 1 = 2 + c \Rightarrow c = -4$$

$$-e^{-y} - e^y = x - 4$$

$$e^y + e^{-y} = 4 - x$$

[5]

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4. $x^2 \frac{dy}{dx} - y^2 = 0$

$$x^2 \frac{dy}{dx} = y^2$$

$$\int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx$$

$$-\frac{1}{y} = -\frac{1}{x} + c$$

$$\frac{1}{y} = \frac{1}{x} - c = \frac{1 - cx}{x}$$

$$y = \frac{x}{1 - cx}$$

An equivalent solution is

$y = \frac{x}{1 + kx}$, since $-c$ can be replaced by k .

[4]

5. (i) $\frac{dC}{dt} = kC$

$$\int \frac{1}{C} dC = \int k dt$$

$$\ln C = kt + c$$

When $t = 0$, $C = 100 \Rightarrow \ln 100 = c$

$$\ln C = kt + \ln 100$$

$$\ln \frac{C}{100} = kt$$

$$\frac{C}{100} = e^{kt}$$

$$C = 100e^{kt}$$

When $t = 2$, $C = 500 \Rightarrow 500 = 100e^{2k} \Rightarrow e^{2k} = 5 \Rightarrow e^k = \sqrt{5}$

$$C = 100e^{kt} = 100(e^k)^t = 100(5^{\frac{1}{2}})^t$$

$$C = 100 \times 5^{\frac{1}{2}t}$$

[7]

(ii) When $t = 5$, $C = 100 \times 5^{2.5} = 5590$ (to nearest whole number).

[2]

(iii) $100 \times 5^{\frac{1}{2}t} > 5000$

$$5^{\frac{1}{2}t} > 50$$

$$\frac{1}{2}t \ln 5 > \ln 50$$

$$t > 4.86$$

After 4.86 seconds.

[3]

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$$(iv) 100 \times 5^{\frac{1}{2}t} > 150$$

$$5^{\frac{1}{2}t} > 1.5$$

$$\frac{1}{2}t \ln 5 > \ln 1.5$$

$$t > 0.504$$

After 0.504 seconds.

[3]

$$6. (i) \int \pi x^{\frac{3}{2}} dx = \int -50 dt$$

$$\frac{2}{5} \pi x^{\frac{5}{2}} = -50t + c$$

$$\text{When } t = 0, x = 25 \Rightarrow \frac{2}{5} \pi \times 25^{\frac{5}{2}} = c \Rightarrow c = 1250\pi$$

$$\frac{2}{5} \pi x^{\frac{5}{2}} = 1250\pi - 50t$$

$$x^{\frac{5}{2}} = 3125 - \frac{125}{\pi} t$$

$$x = \left(3125 - \frac{125}{\pi} t \right)^{\frac{2}{5}}$$

[5]

$$(iii) \text{ When } x = 0, \frac{125}{\pi} t = 3125 \Rightarrow t = 25\pi = 78.5 \text{ seconds.}$$

It takes 78.5 seconds (3 s.f.)

[2]